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H_g-Normal and H_g-Regular Spaces

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ABSTRACT: $\mathcal{H}_g$-normal and $\mathcal{H}_g$-regular spaces are introduced by means of a generalized topological space (X, μ) and a hereditary class $\mathcal{H}$and different characterizations and properties ofthese spaces are discussed.

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I. INTRODUCTION

Let X be a non empty set and *exp*X be the power set of X. A collection $\mu \subset$ expX is called a generalized topology on X if $\emptyset \in \mu$ and μ is closed for arbitrary unions [1]. (X, μ) is called a generalized topological space and the members of $\mu$are called μ-open sets and their complements are called μ-closed sets. An ideal $\mathcal{J}$ on X is a non empty family of subsets of X satisfying (i) A $\subset$ B, B$\in$ $\mathcal{J}$implies A $\in \mathcal{J}$; (ii) A, B $\in \mathcal{J}$ implies A $\cup$ B $\in \mathcal{J}$. If τ is a topologyon X and $\mathcal{J}$is an ideal on X, then (X,, $\mathcal{J}$) is called an ideal space. Jankovic and Hamlett [4] have introduced another topology called *-topology by using a given ideal $\mathcal{J}$ and a given topology τ on X, which is finer thanτ. In [3], Csaszar introduced hereditary classes. A non empty family $\mathcal{H}$ of subsets of X is called a hereditary class if it satisfies only condition (i) of ideals, i.e. A $\subset$ B, B $\in \mathcal{H}$ implies A $\in \mathcal{H}$. Then he introduced an operator ()*: *exp*X $\rightarrow$*exp*X, using a given generalized topology $\mu$and a hereditary class $\mathcal{H}$ on X. He defined another operator c*: *exp*X $\rightarrow$*exp*X, using the operator ()* by c*(A) = A $\cup$A* for A $\subset$ X, which is monotonic, enlarging and idempotent. This operator c* induces another generalized topologycalled μ*-generalized topology, which is finer than μ. The members ofμ* are called $*-$open sets and their complements are called $*-$closed sets.

In [6], the authors have introduced the notions of $\mathcal{J}_g$-normal and $\mathcal{J}_g$-regular spaces using $\mathcal{J}_g$-open sets (A $\subset$ X is called $\mathcal{J}_g$-closed if A*$\subset$U whenever U is open and A $\subset$ U and complements of $\mathcal{J}_g$-closed sets are $\mathcal{J}_g$-open). They have studied the characterizations and properties of such spaces.

The purpose of this paper is to introduce the concepts of $\mathcal{H}_g$-normal and $\mathcal{H}_g$-regular hereditary spaces using $\mathcal{H}_g$-open sets to investigate whether the characterizations and properties of $\mathcal{J}_g$-normal and $\mathcal{J}_g$-regular spaces remain valid by dropping some conditions of topology to form generalized topology and ideal to hereditary classes.

Let (X, μ) be a generalized topological space and$\mathcal{H}$ be a hereditary class on X, then (X, $\mu, \mathcal{H}$) is called hereditary space. If A $\subset$X, $cl_\mu(A)$ and $int_\mu(A)$ will denote the μ-closure and μ-interior of set A in generalized topological space (X, μ) respectively and $cl^*(A)$ and $int^*(A)$ will denote the μ*-closure and μ*-interior of set A in generalized topological space (X, μ*), respectively. A$\subset$ X is called $\mathcal{H}_g$-closed if A$*$ $\subset$ U whenever U is μ-open and A $\subset$ U and complement of $\mathcal{H}_g$-closed set is called $\mathcal{H}_g$-open. A subset A of a generalized topological space (X,μ) is called g-μ-closed if $cl_\mu(A)$ $\subset$ U whenever U is μ-open and A $\subset$ U and complement of g-μ-closed set is called g-μ-open.

II. $\mathcal{H}_g$-NORMAL AND $g\mathcal{H}$-NORMAL SPACES

Definition 1. A hereditary space (X, $\mu, \mathcal{H}$) is said to be $\mathcal{H}_g$-normal if for each pair of disjoint μ-closed sets A and B, there existsdisjoint $\mathcal{H}_g$-open sets U and V such that A $\subset$ U and B $\subset$ V .

Since every μ-open set is $\mathcal{H}_g$-open set, every μ-normal is $\mathcal{H}_g$-normal. But the converse need not be true, shown as in the following example:

Example 2. Let X = {a, b, c}, μ= {$\emptyset$,{a},{a, b},{a, c}, X} and $\mathcal{H}$ = {$\emptyset$,{a}}. Then $\emptyset$* = $\emptyset$, ({a, b})* = {b}, ({a, c})* = {c},({a})* = $\emptyset$ and X* = {b, c}. Every μ-open set is *-closed andtherefore every subset of X is $\mathcal{H}_g$-open. This means (X,$\mu, \mathcal{H}$) is$\mathcal{H}_g$-normal. Also {b} and {c} are disjoint μ-closed sets which arenotseparated by disjoint μ-open sets and therefore (X, $\mu, \mathcal{H}$) is not μ-normal.

Theorem 3. Let (X, $\mu, \mathcal{H}$) be a hereditary space. Then thefollowing are equivalent:

1. X is $\mathcal{H}_g$-normal.
2. For each pair of disjoint μ-closed sets A and B, there existdisjoint $\mathcal{H}_g$-open sets U and V such that A $\subset$ U and B $\subset$ V.
3. For each μ-closed set A and a μ-open set U containing A, thereexists an $\mathcal{H}_g$-open set V such that A $\subset$ V $\subset$ cl^*(V) $\subset$ U.

Proof. 1 ⇒ 2. follows from the definition. 2 ⇒ 3. Let A be aμ-closed set and U be a μ-open set containingA. Then A and X- U are disjoint μ-closed sets, there exist disjoint$\mathcal{H}_g$-open sets V and W such that A ⊂ V and X - U ⊂ W. AlsoV ∩W = ∅ implies that V ∩ int^*(W) = ∅ and therefore cl^*(V) ⊂ X- int^*(W). Since X - U is μ-closed and W is $\mathcal{H}_g$-open, X -U ⊂ W,X - U ⊂ int^*(W), X - int^*(W) ⊂ U. Therefore A ⊂ V ⊂ $cl^*(V)$ ⊂X - int^*(W) ⊂ U.

3 ⇒ 1. Let A and B be any two disjoint μ-closed sets in X. Thenthere exist an $\mathcal{H}_g$-open set V such that A ⊂ V ⊂ cl^*(V) ⊂ X - B.Let W = X - cl^*(V), then V and W are disjoint $\mathcal{H}_g$-open sets suchthat A ⊂ V and B ⊂ W which proves that X is $\mathcal{H}_g$-normal.

Theorem 4. Let (X, $\mu,\mathcal{H}$) be an $\mathcal{H}_g$-normal space. If F is μ-closed and A is a g- μ-closed set such that A ∩ F =∅, then thereexist disjoint $\mathcal{H}_g$-open sets U and V such that A ⊂ U and F ⊂ V .

Proof. A ∩ F = ∅, A ⊂ X - F, where X - F is μ-open. Socl_μ(A) ⊂ X - F. cl_μ(A) ∩ F = ∅ and X is $\mathcal{H}_g$-normal, thereexist disjoint $\mathcal{H}_g$-open sets U and V such that cl_μ(A) ⊂ U andF ⊂ V.

The following corollary gives characterization of μ-normal spaces.If we take $\mathcal{H}$ = {∅} in above theorem, then we have the corollarybelow:

Corollary 5. Let (X, μ) be a μ-normal space. If F is μ-closedand A is a g-μ-closed set such that A ∩ F = ∅, then there existdisjoint g-μ-open sets U and V such that A ⊂ U and F ⊂ V .

Theorem 6. Let (X, $\mu,\mathcal{H}$) be a hereditary space which is $\mathcal{H}_g$-normal. Then the following conditions hold:

1. For each μ-closed set A and each g-μ-open set B containing A,there exists an $\mathcal{H}_g$-open set U such that A ⊂ int^*(U) ⊂ U ⊂ B.
2. For each g-μ-closed set A and each μ-open set B containing A,there exists an $\mathcal{H}_g$-closed set U such that A ⊂ U ⊂ cl^*(U) ⊂ B.

Proof. 1. Let A be a μ-closed set and B be a g-μ-open setcontaining A. Then A ∩(X - B) = ∅ where A is a μ-closed set andX - B is g-μ-closed set. By above theorem, there exist disjoint $\mathcal{H}_g$-open sets U and V such that A ⊂ U and X − B ⊂ V . Since U andV are disjoint, U ⊂ X - V . Also A ⊂ int^*(U), since A is μ-closedset and A ⊂ U. Therefore, A ⊂ int^*(U) ⊂ U ⊂ X - V ⊂ B.

2. Let A be a g-μ-closed set and B be a μ-open set containing A.Then X - B ⊂ X - A, where X - A is g-μ-open set and X - B is μ-closed set. By 1., there exists an $\mathcal{H}_g$-open set V such that X - B ⊂ int^*(V) ⊂ V ⊂ X - A. Therefore, A ⊂ X - V ⊂ cl^*(X - V) ⊂ B.If we take U = X - V, then A ⊂ U ⊂ cl^*(U) ⊂ B and U is an$\mathcal{H}_g$-closed set.

The following corollary gives characterization of μ-normal spaces.If we take $\mathcal{H}$ = {∅} in above theorem, then we have the corollary below:

Corollary 7. Let (X, μ) be a μ-normal space. Then the following conditions hold:

1. For each μ-closed set A and each g-μ-open set B containing A,there exists a g-μ-open set U such that A ⊂ int_μ(U) ⊂ U ⊂ B.
2. For each g-μ-closed set A and each μ-open set B containing A,there exists a g-μ-closed set U such that A ⊂ U ⊂ cl_μ(U) ⊂ B.

Definition 8. A hereditary space (X, $\mu,\mathcal{H}$) is said to be g- $\mathcal{H}$-normal if for each pair of disjoint $\mathcal{H}_g$-closed sets A and B, thereexists disjoint μ-open sets U and V such that A ⊂ U and B ⊂ V .

Since every μ-closed set is $\mathcal{H}_g$-closed, every g- $\mathcal{H}$-normal spaceis μ-normal. But the converse need not be true shown as in the following example:

Example 9. Let X = {a, b, c, d}, μ = {∅,{a},X} and $\mathcal{H}$ = {∅,{a}}. Then ({a})* =∅. Every μ-open set is *-closed and therefore every subset of X is $\mathcal{H}_g$-closed. {a, b} and {c, d} are disjoint$\mathcal{H}_g$-closed sets which are not separated by disjoint μ-open sets, so (X, $\mu,\mathcal{H}$) is not g- $\mathcal{H}$-normal. Also there is no pair of disjoint μ-closed sets, so (X, $\mu,\mathcal{H}$) is obviously μ-normal.

Theorem 10. Let (X, $\mu,\mathcal{H}$) be a hereditary space. Then the following are equivalent:

1. X is g- $\mathcal{H}$-normal.
2. For each $\mathcal{H}_g$-closed set A and an $\mathcal{H}_g$-open set B containing A,there exists a μ-open set V such that A ⊂ V ⊂ cl_μ(V) ⊂ B.

Proof. 1 ⇒ 2. Let A be an $\mathcal{H}_g$-closed set and B be an $\mathcal{H}_g$-openset containing A. Then A and X - B are disjoint $\mathcal{H}_g$-closed sets,there exist disjoint μ-open sets V and W such that A ⊂ V andX - B ⊂ W. Also V ∩ W = ∅ implies that cl_μ(V) ⊂ X - W. Therefore A ⊂ V ⊂ cl_μ(V) ⊂ X -W ⊂ B.

2 ⇒ 1. Let A and B be any two disjoint $\mathcal{H}_g$-closed sets in X. ThenA ⊂ X - B, where X - B is $\mathcal{H}_g$-open set. Then there exist μ-open setV such that A ⊂ V ⊂ cl_μ(V) ⊂ X - B. Let W = X - cl_μ(V), thenV and W are disjoint μ-open sets such that A ⊂ V and B ⊂ W.Therefore (X, $\mu,\mathcal{H}$) is g- $\mathcal{H}$-normal.

If we take $\mathcal{H}$= {∅} in the above theorem, then we have the following characterization of g-μ-normal spaces.

Corollary 11. Let (X, μ) be a generalized topological space.Then the following are equivalent:

1. X is g-μ-normal.
2. For each g-μ-closed set A and each g-μ-open set B containingA, there exists a μ-open set U such that A ⊂ U ⊂ cl_μ(U) ⊂ B.

Theorem 12. Let (X, $\mu,\mathcal{H}$) be a hereditary space. Then the following are equivalent:

1. X is g-$\mathcal{H}$-normal.

2. For each pair of disjoint $\mathcal{H}_g$-closed sets A and B, there exists a μ-open set V containing A such that cl_μ(V) ∩ B = ∅.
3. For each pair of disjoint $\mathcal{H}_g$-closed sets A and B, there exists aμ-open set U containing A and a μ-open set V containing B suchthat cl_μ(U) ∩ cl_μ(V) = ∅.
Proof. 1⇒ 2. Let A and B be disjoint $\mathcal{H}_g$-closed sets in X.Then A ⊂ X - B, where X - B is $\mathcal{H}_g$-open set. There existsa μ-open set V such that A ⊂ V ⊂ cl_μ(V) ⊂ X - B, and socl_μ(V) ∩ B = ∅.
2 ⇒ 3. Let A and B be disjoint $\mathcal{H}_g$-closed sets in X. Then thereexists a μ-open set U such that A ⊂ U and cl_μ(U) ∩ B = ∅. Nowcl_μ(B) and B are disjoint $\mathcal{H}_g$-closed sets in X. Therefore thereexists a μ-open set V such that B ⊂ V and cl_μ(U) ∩ cl_μ(V) = ∅.
3⇒1. obviously true.
The following corollary gives a characterization of g-μ-normalspaces if we take $\mathcal{H}$ = {∅} in the above theorem.
Corollary 13. Let (X,μ) be a generalized topological space.Then the following are equivalent:
1. X is g-μ-normal.
2. For each pair of disjoint g-μ-closed sets A and B, there exists aμ-open set V containing A such that cl_μ(V) ∩ B = ∅.
3. For each pair of disjoint g-μ-closed sets A and B, there exists aμ-open set U containing A and a μ-open set V containing B such that cl_μ(U) ∩ cl_μ(V) = ∅.
Theorem 14. Let (X, $\mu, \mathcal{H}$) be a g-$\mathcal{H}$-normal space. If A andB are disjoint $\mathcal{H}_g$-closed sets in X, then there exists disjoint μ-opensets U and V such that cl^*(A) ⊂ U and cl^*(B) ⊂ V.
Proof. Let A and B be disjoint $\mathcal{H}_g$-closed sets in X. Then thereexists a μ-open set U containing A and a μ-open set V containingB such that cl_μ(U) ∩ cl_μ(V) = ∅. Also A is $\mathcal{H}_g$-closed, A ⊂ U,therefore cl^*(A) ⊂ U. Similarly cl^*(B) ⊂ V.

Taking $\mathcal{H}$= {∅} in the above theorem, gives a property of g-μ-normal spaces as shown in the corollary:
Corollary 15. Let (X, μ) be a g-μ-normal space. If A and Bare disjoint g-μ-closed sets in X, then there exists disjoint μ-opensets U and V such that cl_μ(A) ⊂ U and cl_μ(B) ⊂ V.
Theorem 16. Let (X, $\mu, \mathcal{H}$) be a g-$\mathcal{H}$-normal space. If A isan $\mathcal{H}_g$-closed set and B is an $\mathcal{H}_g$-open set containing A, then thereexists a μ-open set U such that A ⊂ cl^*(A) ⊂ U ⊂ int^*(B) ⊂ B.
Proof. Let A be an $\mathcal{H}_g$-closed set and B be an $\mathcal{H}_g$-open setcontaining A. Since A and X - B are disjoint $\mathcal{H}_g$-closed sets, thereexist disjoint μ-open sets U and V such that cl^*(A) ⊂ U and cl^*(X - B) ⊂ V . Now X - int^*(B) = cl^*(X - B) ⊂ V implies that X - V ⊂ int^*(B). Also U ∩ V = ∅, U ⊂ X - V and so A ⊂ cl^*(A) ⊂ U ⊂X - V ⊂ int^*(B) ⊂ B.
Taking $\mathcal{H}$ = {∅} in the above theorem, gives a property of g-μ-normal spaces as shown in the corollary:
Corollary 17. Let (X, μ) be a g-μ-normal space. If A is ag-μ-closed set and B is a g-μ-open set containing A, then thereexists a μ-open set U such that A ⊂ cl_μ(A) ⊂ U ⊂ int_μ(B) ⊂ B.
The following theorem gives a characterization of μ-normal spacesin terms of g-μ-open sets:
Theorem 18. Let (X, μ) be a generalized topological space. Then the following are equivalent:
1. X is μ-normal.
2. For each pair of disjoint μ-closed sets A and B, there exist disjoint g-μ-open sets U and V such that A ⊂ U and B ⊂ V.
3. For each μ-closed set A and a μ-open set V containing A, there exists a g-μ-open set U such that A ⊂ U ⊂ cl_μ(U) ⊂ V.

III. $\mathcal{H}_g$-REGULAR AND $g\mathcal{H}$-REGULAR SPACES

Definition 19. A hereditary space (X, $\mu, \mathcal{H}$) is said to be $\mathcal{H}_g$-regular if for each point x and a μ-closed set B not containing x,there exists disjoint $\mathcal{H}_g$-open sets U and V such that x ∈ U andB ⊂ V.

Since every μ-open set is $\mathcal{H}_g$-open set, every μ-regular is $\mathcal{H}_g$-regular. But the converse need not be true, as shown in the following example:
Example 20. Let X = {a, b, c}, μ = {∅,{a},{a, b},{a, c}, X} and $\mathcal{H}$ = {∅,{a}}. Then ∅* = ∅, ({a, b})* = {b}, ({a, c})* = {c},({a})* = ∅ and X* = {b, c}. Every μ-open set is *-closed and therefore every subset of X is $\mathcal{H}_g$-open. This means (X, $\mu, \mathcal{H}$) is$\mathcal{H}_g$-regular. Also {c} is μ-closed set not containing b and {c} and b are not separated by disjoint μ-open sets and therefore (X, $\mu, \mathcal{H}$) is not μ-regular.
Theorem 21. Let (X, $\mu, \mathcal{H}$) be a hereditary space. Then the following are equivalent:
1. X is $\mathcal{H}_g$-regular.
2. For each μ-closed set B not containing x ∈ X, there existdisjoint $\mathcal{H}_g$-open sets U and V such that x ∈ U and B ⊂ V.
3. For each μ-open set U containing x ∈X, there exists an $\mathcal{H}_g$-openset V such that x ∈ V ⊂ cl^*(V) ⊂ U.
Proof. 1 ⇒ 2. follows from the definition.
2 ⇒ 3. Let U be a μ-open set such that x ∈ U. Then X - U isμ-closed set not containing x. Therefore, there exist disjoint $\mathcal{H}_g$-open sets V and W such that x ∈ V and X - U ⊂ W. ThereforeX - U ⊂ int^*(W) and X -int^*(W) ⊂ U. Also V ∩ W = ∅implies that V ∩ int^*(W) = ∅, cl^*(V) ⊂ X -int^*(W). Thereforex ∈ V ⊂ cl^*(V) ⊂ U.
3 ⇒1. Let B be any μ-closed set not containing x in X. Thenthere exist an $\mathcal{H}_g$-open set V such that x ∈ V ⊂ cl^*(V) ⊂ X - B.

Let W = X - cl^*(V), then V and W are disjoint $\mathcal{H}_g$-open sets suchthat x ∈ V and B ⊂ W, which proves that X is$\mathcal{H}_g$-regular.

The following corollary gives characterization of μ-regular spaces.If we take $\mathcal{H} = \{\emptyset\}$ in above theorem, then we have the corollary below:

Corollary 22. Let (X, μ) be a generalized topological space.Then the following are equivalent:

1. X is μ-regular.
2. For each μ-closed set B not containing x ∈ X, there existdisjoint g-open sets U and V such that x ∈ U and B ⊂ V.
3. For each μ-open set U containing x ∈ X, there exists a g-openset V such that x ∈ V ⊂ cl_μ(V) ⊂ U.

Theorem 23. If every μ-open subset of a hereditary space (X, $\mu, \mathcal{H}$) is *-closed, then (X, $\mu, \mathcal{H}$) is $\mathcal{H}_g$regular.

Proof. Let every μ-open set of X be *-closed. Then every subset of X is $\mathcal{H}_g$-closed and therefore every subset of X is $\mathcal{H}_g$-open.Therefore (X, $\mu, \mathcal{H}$) is $\mathcal{H}_g$-regular.

The following example shows that the converse of the above theorem need not be true:

Example 24. Let X = R with the usual topology μ whichis also a generalized topology and $\mathcal{H} = \{\emptyset\}$. Then X is μ-regularand therefore $\mathcal{H}_g$-regular. But μ-open sets are not μ-closed andtherefore μ-open sets are not *-closed.

The following theorem gives characterizations of μ-regular hereditary spaces where the hereditary class is completely codense.

Theorem 25. Let (X, $\mu, \mathcal{H}$) be a hereditary space, where $\mathcal{H}$is completely codense. Then the following are equivalent:

1. X is μ-regular.
2. For each μ-closed set B not containing x ∈ X, there exist disjoint*-open sets U and V such that x ∈ U and B ⊂ V.
3. For each μ-open set U containing x ∈ X, there exists a *-open set V such that x ∈ V ⊂ cl^*(V) ⊂ U.

Proof. 1 ⇒ 2: Let B be μ-closed set not containing x ∈ X.Since X is μ-regular, there exist disjoint μ-open sets U and V suchthat x ∈ U and B ⊂ V. Every μ-open set is *-open, which proves2.

2 ⇒ 3: Let U be μ-open set containing x ∈ X. Then X - U is μ-closed set not containing x. By 2, there exist disjoint *-open sets V and W such that x ∈ V and X - U ⊂ W. V and W are disjoint,V ⊂ X - W and X - W is *-closed, cl^*(V) ⊂ X - W ⊂ U. HenceV is the required *-open set such that x ∈ V ⊂ cl^*(V) ⊂ U.

3 ⇒ 1: Let B be μ-closed set not containing x ∈ X. Then X - Bis μ-open set containing x. By 3, there exists a *-open set V such that x ∈ V ⊂ cl^*(V) ⊂ X - B. Let U = X -cl^*(V), thenB ⊂ U and U and V are disjoint*-open sets. Since $\mathcal{H}$ is completely codense, therefore every *-open set is μ-α-open. So B ⊂ U ⊂ $int_\mu(cl_\mu(int_\mu$(U))) = G and V ⊂ $int_\mu(cl_\mu(int_\mu$(V))) = H. Then G and H are disjoint μ-open sets such that x ∈ H and B ⊂ G. HenceX is μ-regular.

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